

# The state space of short-range Ising spin glasses: the density of states

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## Abstract

The state space of finite square and cubic Ising spin glass models is analysed in terms of the global and the local density of states. Systems with uniform and gaussian probability distribution of interactions are compared. Different measures for the local state density are presented and discussed. In particular the question whether the local density of states grows exponentially or not is considered. The direct comparison of global and local densities leads to consequences for the structure of the state space.

## 1 Introduction

The often very unusual dynamic behaviour of complex systems like spin glasses [1, 2] is significantly determined by the properties of their state space. One key to understand the relaxation and aging effects in this class of systems in particular for the low-temperature region is given by the structure of local minima and barriers in the low-lying energy landscape. In order to construct models of this landscape, which are useful for simulating the non-equilibrium dynamics, it is necessary to extract and to quantify the important structural properties of this landscape. Unfortunately in experiments the state-space structure is only indirectly accessible. For systems with long-range interactions analytical

mean-field-like methods have been applied to investigate the state-space properties. The situation for short-range systems is more complicated. Due to the computational effort needed the exact calculation of the energetically low-lying excitations of a system is restricted to small system sizes. Nevertheless the analysis of small systems can give a first understanding of effects in principle and help to check model assumptions.

One of the simplest models for complex systems is the Ising-spin glass. There has been done a lot of research concerning the long-range SK-model [3]. A few numerical and experimental works tried to analyse the state-space structure more or less directly [4, 5]. This was mostly done in order to check the interesting theoretical predictions for the hierarchical structure of the phase space of the SK model [6]. For short-range systems the situation is more unsatisfying. For small  $\pm J$  model systems a detailed analysis of the state-space has been done in [7]. It is unclear how strong the state-space structure found is influenced by the discreteness of the interactions. As a counterpart to the  $\pm J$  systems usually systems with gaussian distributed interactions between nearest neighbours are treated. An extensive analysis of the morphology of the state space was undertaken in [8] by use of the so called lid method.

An interesting outcome of this investigation was, that the local density of states inside a state-space pocket seems to grow exponentially. Such an exponential increase of the local density of states with increasing energy would lead to drastic thermodynamic consequences. For temperatures below a critical temperature the occupation probability would reach its maximum at the ground-state energy. Thus the system is trapped in a certain state-space valley for a long time or even forever, provided that the barriers surrounding this pocket are high enough. For increasing temperatures the maximum of the occupation probability jumps at a critical temperature  $T_c$  from the minimal to the maximal energy of the system. Therefore the probability to leave the considered valley increases drastically. The system is no longer trapped in this valley.

This behaviour is not only important from a thermodynamic point of view, but for optimization problems and methods, too [9]. Assuming there exists

such a critical temperature, the cooling scheme for simulated annealing methods should be chosen in such a way, that the algorithm has found the ground-state valley at a temperature above the critical one. Otherwise it may happen, that the algorithm never finds the true ground-state due to the low probability to jump to other valleys below the critical temperature.

If the local density of states is exactly exponential, the critical temperature is sharply defined. However, if there is no exponential behaviour, the transition might vanish or is at least smeared out. In this paper we try to clarify this situation. We analysed finite two- and three-dimensional systems with respect to their density of states. Starting from the exact knowledge of all energetically low-lying states we calculated at first the global density of states. After sorting the states according to the valley in state space they belong to, we will discuss various differently measures for the local state density. Finally we will compare these different measures.

## 2 Model and methods

In the following we will present results for two- and three-dimensional Ising-spin systems on square and cubic lattices with randomly chosen interactions between nearest neighbours and periodic boundary conditions. The lattice size is restricted by computational reasons and is  $L = 8$  for the two-dimensional and  $L = 4$  for the three-dimensional case. There is no external field applied to the systems.

We analysed systems with a gaussian distribution of interactions as well as systems with a uniform distribution. A disadvantage of the gaussian distribution of interactions in particular for local structure investigations of the state space is the possibility of extremely large local fields. These fields can lead to a crossing of all energy barriers by just a single spin flip. This unphysical drawback can be overcome by using a uniform distribution, which is in this sense a counterpart to the gaussian one. It restricts the maximal strengths of interactions and thus the maximal local field.

In order to allow the comparison of both distributions the first two moments

have been set equal. As usual the mean is set to zero and the standard deviation is normalized to unity. If this choice leads to a very similar state-space structure, both system classes could be used alternatively.

The basis of the state-space analysis is an exact determination of all energetically low-lying states up to a given cut-off energy by the method of recursive branch-and-bound [10]. The main idea of this method is to search the binary tree of all states. The search can be restricted by finding lower bounds for the minimal reachable energy inside of a subtree. If this lower bound is higher than the energy of a suboptimal state already found, it is not necessary to examine the corresponding subtree. A first good suboptimal state can be found by recursively solving smaller subproblems. By adding an energy offset to the calculated lower bounds it is possible to calculate not only the ground states, but all states below a given cut-off energy too.

The obtained states were ordered by increasing energy using a distributed sort algorithm. Starting from the ground state and successively increasing the maximal energy of the considered states all states are sorted according to their valley in the configuration space. For a chosen energy two states are sorted to the same valley, if one state can be reached from the other via a series of single spin flips without exceeding the chosen energy. Thus the definition of a valley depends on this energy. Each valley can be addressed by the state with minimal energy inside of the valley. Note, that a valley is joined with a more low-lying valley, if the considered energy becomes larger than the barrier between both valleys. Furthermore it should be noted here, that the definition of a valley of course depends on the definition of neighbouring spin configurations. As it is done in most investigations we restricted ourself to consider only single spin flip processes.

### 3 Results

The global density of states  $g_{\text{global}}(\varepsilon)$  (GDOS) is defined as the number of states with energy  $\varepsilon$  per spin above the ground state. Fig. 1 shows the logarithm of the global density of states normalized to the number of spins  $N$  in the system

for the  $2d$  and the  $3d$  systems with gaussian and uniform distribution. The results are averaged over 20 different realizations of disorder for the gaussian distribution and over 50 samples for the uniform distribution. The errorbars give an idea of the sample to sample fluctuations.

The GDOS is significantly higher for the  $2d$  systems compared to the  $3d$  systems. This is obviously caused by the different coordination numbers, as can be seen in fig. 2. The GDOS for the gaussian distribution is slightly higher than for the uniform distribution. Nevertheless, there seems to be no qualitative difference between both curves.

For all systems the GDOS increases clearly subexponentially with energy. To quantify this behaviour it is possible to make an ansatz of the form

$$g(\varepsilon) \propto \exp\left(c + \alpha\varepsilon + \gamma\varepsilon^\delta\right) \quad (1)$$

for small energies  $\varepsilon$  above the minimal energy. The occupation probability in equilibrium then reads

$$p(\varepsilon) \propto \exp\left[c + (\alpha - \beta)\varepsilon + \gamma\varepsilon^\delta\right], \quad (2)$$

where  $\beta$  denotes the inverse temperature. The extremal value of such a distribution is reached for

$$\varepsilon_{\text{ext}} = \left(\frac{\beta - \alpha}{\gamma\delta}\right)^{1/(\delta-1)}. \quad (3)$$

The only singular point in (3) is at  $\beta = \alpha$ . In the linear case  $\gamma = 0$  the maximum of (2) jumps at this value of  $\beta$  from the maximal energy of the system for high temperatures to the minimal energy for low temperatures.

If however  $\gamma \neq 0$  and  $\delta > 1$  the subexponential behaviour of the DOS as found in our data leads to a negative coefficient  $\gamma$ . Then it can easily be seen that the energy of the maximum of (2) is positive and finite for high temperatures and goes down with decreasing temperature. At and below  $T = 1/\alpha$  the occupation probability is highest for  $\varepsilon_{\text{ext}} = 0$ . If the linear term in  $g(\varepsilon)$  vanishes ( $\alpha = 0$ ) and  $0 < \delta < 1$ , the maximum energy goes continuously from the maximal energy of the system down to the minimal one with decreasing temperature. Thus there is no sign of a critical behaviour caused by the DOS.

As a result of the above discussion we chose two different ansatzes for fitting functions in order to analyse our numerical data. For  $\delta = 2$  (1) simplifies to a quadratic polynomial ansatz, which we call in the following the quadratic fit. The choice  $\alpha = 0$  leads to a fitting ansatz without any linear term, which will be called power fit.

The quadratic fits shown in fig. 1 fit the data quite well. The ratio of the linear and the quadratic coefficient corresponds for the  $3d$  systems to an energy of about 0.47 per spin. The inverse linear coefficients correspond to a temperature of  $T \sim 0.5$  in  $2d$  and  $T \sim 0.65$  in  $3d$ . The errors of these fit parameters have been estimated. For the energy per spin it is of the order 0.05 per spin, the error for the temperature can be estimated to 0.05. The lines in fig. 2 are power fits, which seem to fit the data quite well, too. The gaussian and the uniform distribution differ only in the coefficients  $c$  and  $\gamma$ . The exponents  $\delta$  are about 0.72 for both distributions. The error is of the order 0.05.

The local DOS (LDOS) is given by the number of states inside a valley at a given energy. In order to average the LDOS, it is necessary to clarify the measuring procedure. We discuss here three different possibilities.

The first one is to start at a high temperature and to perform a steepest descent algorithm. The LDOS of the valley the system was trapped in, can then be measured relatively to the minimal energy of this valley. The averaging will be done over different runs and different realizations of interactions. We will call the measure defined in that way relatively measured LDOS (RLDOS) and denote it by  $g_{\text{rel}}$ .

The second possibility assumes that the ground-state of the system is known already. Then the local density of states can be measured relatively to the ground-state energy instead of the minimal energy of the valley found. We will call this variant absolutely measured LDOS (ALDOS) and denote it by  $g_{\text{abs}}$ . If the averaging procedures for  $g_{\text{rel}}$  and  $g_{\text{abs}}$  are restricted to the ground-state valley, both variants are equivalent and result in the averaged local density of ground-state valleys (GLDOS), which will be denoted by  $g_{\text{gs}}$ .

It should be noted here, that in practice the averaging will be performed

not over different runs, but over all valleys found up to the cut-off energy. This may cause an systematic error due to valleys with local minima higher than the cut-off energy. Obviously, this effect could only be important for  $g_{\text{rel}}$ .

As for the GDOS the RLDOS as a function of the energy per bond  $\varepsilon/d$  is quite equivalent for  $2d$  and  $3d$  systems (Fig. 3). However, for energies higher than 0.02 per bond there seem to be systematic deviations. In both dimensions  $g_{\text{rel}}$  is slightly higher for the gaussian distribution. Both fitting ansatzes fit the numerical data quite well, as can be seen by the examples given in fig. 3. The linear coefficients of the quadratic fits correspond to critical temperatures of about 0.85 in the  $3d$  case. The ratio between the linear and the quadratic coefficients is equivalent to an energy of about 0.6 ( $2d$ ) and 0.9( $3d$ ). The alternative power fit results in an exponent  $\delta \sim 0.85$  for both distributions.

For the absolutely measured DOS it is not possible to map the results for  $2d$  to the results in  $3d$  by taking into account the different coordination numbers (Fig. 4). The planar systems result in a lower ALDOS compared to the cubic systems. Moreover the power fits lead to exponents  $\delta$ , which are very close to unity. The only exception is the  $2d$  uniform distributed system with  $\delta \sim 1.17$ . The quadratic fits result in ratios between the linear and the quadratic coefficients larger than 1.0 per spin ( $2d$  gaussian) and larger than 2.0 per spin ( $3d$ ), which is almost the inverse ground-state energy per spin. The exception is again the  $2d$  uniform distributed system with a ratio of about 0.3 per spin. All in all the ALDOS grows almost exponentially with energy and the linear coefficients correspond to temperatures of about 0.82 in  $3d$  and 0.71 or 0.87 for  $2d$  systems with gaussian or uniform distribution, respectively.

For the averaged LDOS of the ground-state valleys the  $2d$  uniform distributed case seems to be an exception, too (Fig. 5). It is not clear, whether this is really an effect or just a problem of the statistical errors. The  $g_{\text{gs}}$  versus energy per bond curves for the other cases agree quite well. The power fits lead to an exponent  $\delta$  between 0.77 and 0.89. According to the ratio of the linear and the quadratic term of the quadratic fits the deviations from the linear behaviour are of the order unity for energies between 0.6 and 0.9 per spin. The

linear terms correspond to temperatures of about 0.5 ( $2d$ ) and 0.65 ( $3d$ ), which agree with the GDOS values.

In fig. 6 the different DOS measures are compared for the  $3d$  uniform distributed case. With increasing energy all valleys are joined successively with more low-lying valleys. If there is essentially only the ground-state valley left, the different DOS measures become equivalent. This seems to be the case at an energy of about 0.13 per spin. For all lower energies the global DOS, which counts the states in all existing valleys, is certainly larger than the DOS of the ground-state valleys. Because  $g_{abs}$  is averaged over the ground-state valley and more high-lying valleys and  $g_{abs}$  is always lower than the ground-state valley DOS  $g_{gs}$ , for absolutely measured energies the local DOS of the high-lying valleys is smaller than the GLDOS. On the other hand the relatively measured RLDOS  $g_{rel}$ , which measures the DOS relatively to the minimal energy of a valley, is always larger than the GLDOS. Therefore the more high lying valleys must have larger local densities of states than the ground-state valley measured relatively to the minimal energy of these valleys.

## 4 Summary

We investigated the global and the local DOS for square and cubic Ising spin glass systems with a gaussian and with a uniform probability distribution of interactions, respectively. The quantitative differences between the  $2d$  and the  $3d$  systems are mostly caused by their different coordination numbers. Although the first two moments of the chosen distributions of interactions were set equal, the DOS for the gaussian systems is slightly higher than for the uniform distributed systems. However there is no significant qualitative difference. Therefore it should be possible to use both distributions alternatively for investigations of the state-space structure.

From the direct comparison of the different DOS measures it follows, that at a given absolute energy most of the valleys have a lower LDOS than the ground-state valley. On the other hand, the more high lying valleys have a higher LDOS measured relatively to the minimal energy of the considered valley. Thus we

get a picture of the state space with small energetically low-lying valleys which have high energy barriers and wide energetically high-lying valleys with low energy barriers.

The existence of a large ground-state valley in the system could explain, why simple heuristic and approximative optimizing algorithms often are able to find very good sub-optimal states in problems of this kind. In a first approximation the probability to find the ground-state valley of a system by a random search at a given energy is defined by the ratio between the DOS of the ground-state valley and the global DOS. In fig. 6 this ratio is for high energies close to unity and decreases for lower energies. Therefore a simple search algorithm can easily find the true ground-state valley at high energies (or high temperatures). With decreasing energy or temperature the chance to hit the right sub-valley decreases. Thus the algorithm will find sub-optimal solutions, but not necessarily the optimal state.

A second feature of the state space picture seen here is, that for valleys which start at a high energy the RL DOS grows faster with energy than for valleys which start at a lower energy. As the LDOS should determine most of the non-equilibrium thermodynamic properties seen in real or computer experiments, these properties will depend on the energy range at which the system is investigated. This should be kept in mind while approximating ground-state or low-temperature properties by the investigation of energetically high-lying valleys.

The more detailed quantitative analysis of the DOS shows that only the absolutely measured local DOS  $g_{\text{abs}}$  grows almost exponentially. All other measures for the local DOS and the global DOS grow clearly subexponentially. In all these cases the applied two trial fits with a quadratic and a power ansatz, respectively, describe the numerical data for the logarithm of the DOS quite well.

For the quadratic fits the corrections to the linear behaviour become of the order unity for energies of about 0.5 per spin. The linear coefficients correspond to temperatures, which are for the  $3d$  systems in the region of the transition

temperature found for the gaussian systems [11]. However, although there should only be a zero-temperature transition in  $2d$ , the temperatures resulting from the quadratic fits are about 0.5. The power fits differ in the absolute terms and the coefficients of the power terms. The exponents are with values about 0.7 significantly different to a linear behaviour. Therefore there is at least no sharply defined transition temperature, below which a system is trapped in a valley.

The question whether the form of the local DOS leads to a transition at all remains unclear. To decide this, a further analysis of the occupation probability of a valley with respect to the distribution of energy barriers of this valley would be necessary. Furthermore the barrier distribution combined with the density of local minima will give a better understanding of the physical meaning and the connections between the different LDOS measures.

It should be noted here that the definition of the different DOS measures does not depend on the underlying model. Therefore the quantitative analysis of these properties should give a better insight in the state space structure of similar physical and optimization problems, too.

## References

- [1] J. Hammann, M. Lederman, M. Ocio, R. Orbach and E. Vincent, Physica A **185** (1992) 278.
- [2] J.-O. Andersson, J. Mattsson and P. Nordblad, Phys. Rev. B **48** (1993) 13977.
- [3] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **32** (1975) 1792.
- [4] K. Nemoto, J. Phys. A **21** (1988) L287.
- [5] D. Vertechi and M. A. Virasoro, J. Phys. **50** (1989) 2325.
- [6] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse and M. A. Virasoro, Phys. Rev. Lett. **52** (1984) 1156.

- [7] T. Klotz and S. Kobe, Acta Phys. Slovaca **44** (1994) 347.
- [8] P. Sibani and P. Schriver, Phys. Rev. B **49** (1993) 6667; P. Sibani and J. O. Andersson, Physica A **206** (1994) 1.
- [9] J. C. Schön , J. Phys. A **30** (1997) 2367.
- [10] A. Hartwig, F. Daske and S. Kobe, Comp. Phys. Commun. **32** (1984) 133.
- [11] R. N. Bhatt and A. P. Young, Phys. Rev. B **37** (1988) 5606.

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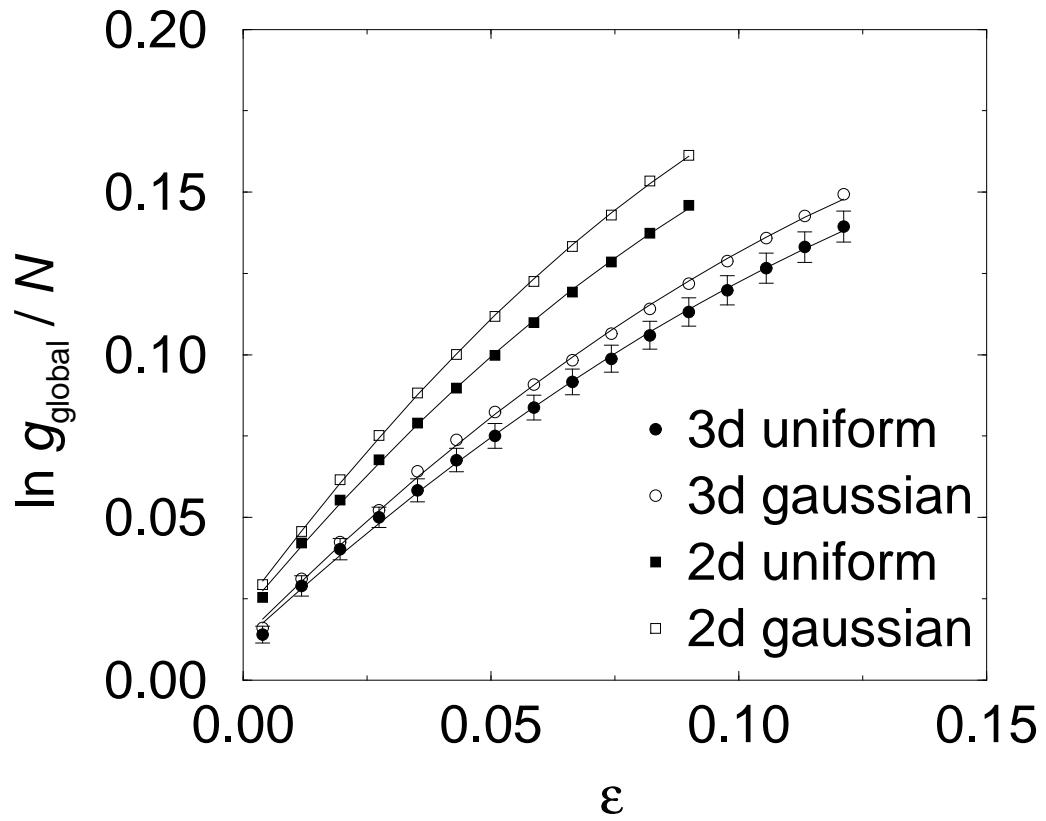


Figure 1: Global density of states  $g_{\text{global}}(\varepsilon)$  for  $2d$  (squares) and  $3d$  (circles) systems with uniform (full symbols) or gaussian (open symbols) distribution of interactions averaged over different realizations of interactions. The lines correspond to quadratic fits.

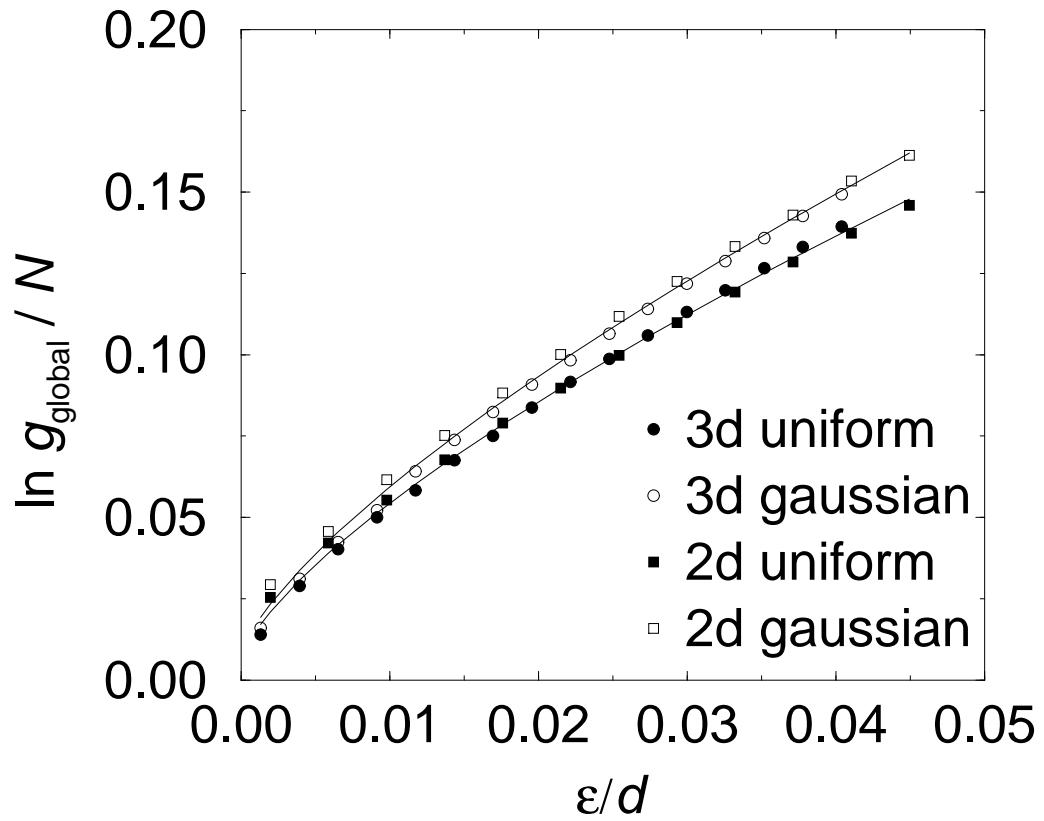


Figure 2: Global density of states  $g_{\text{global}}$  versus energy per bond for 2d (squares) and 3d (circles) systems with uniform (full symbols) or gaussian (open symbols) distribution of interactions averaged over different realizations of interactions. The lines correspond to power fits.

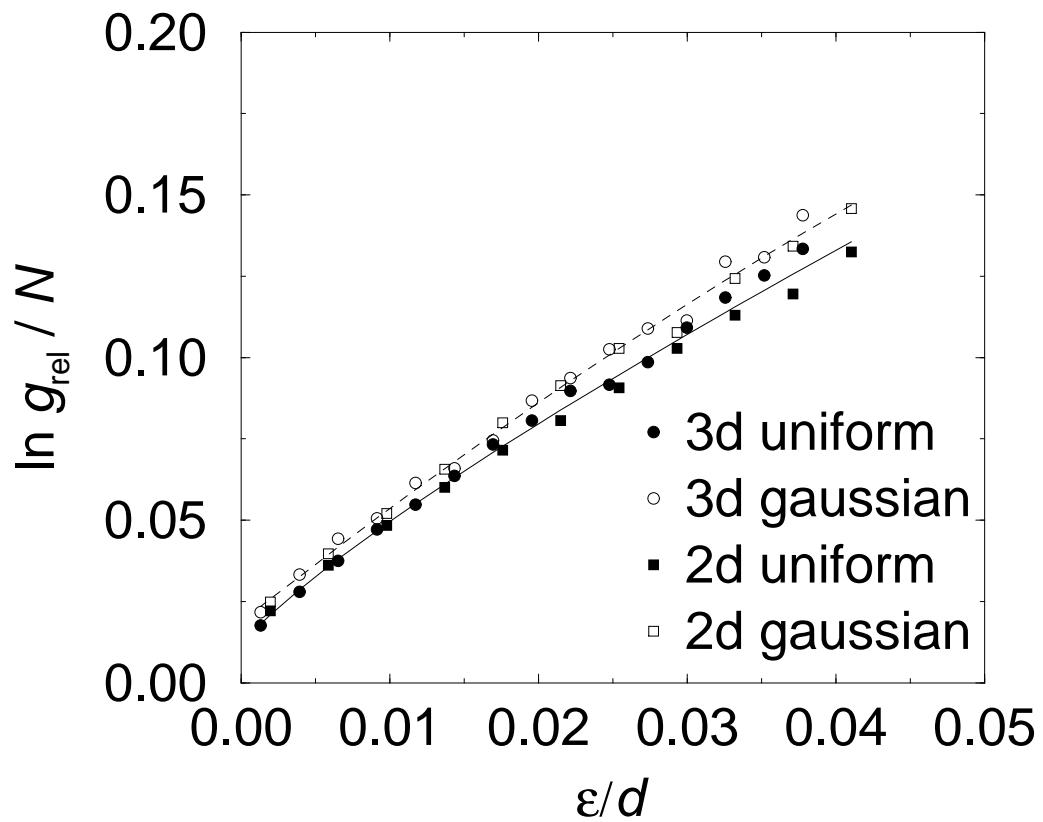


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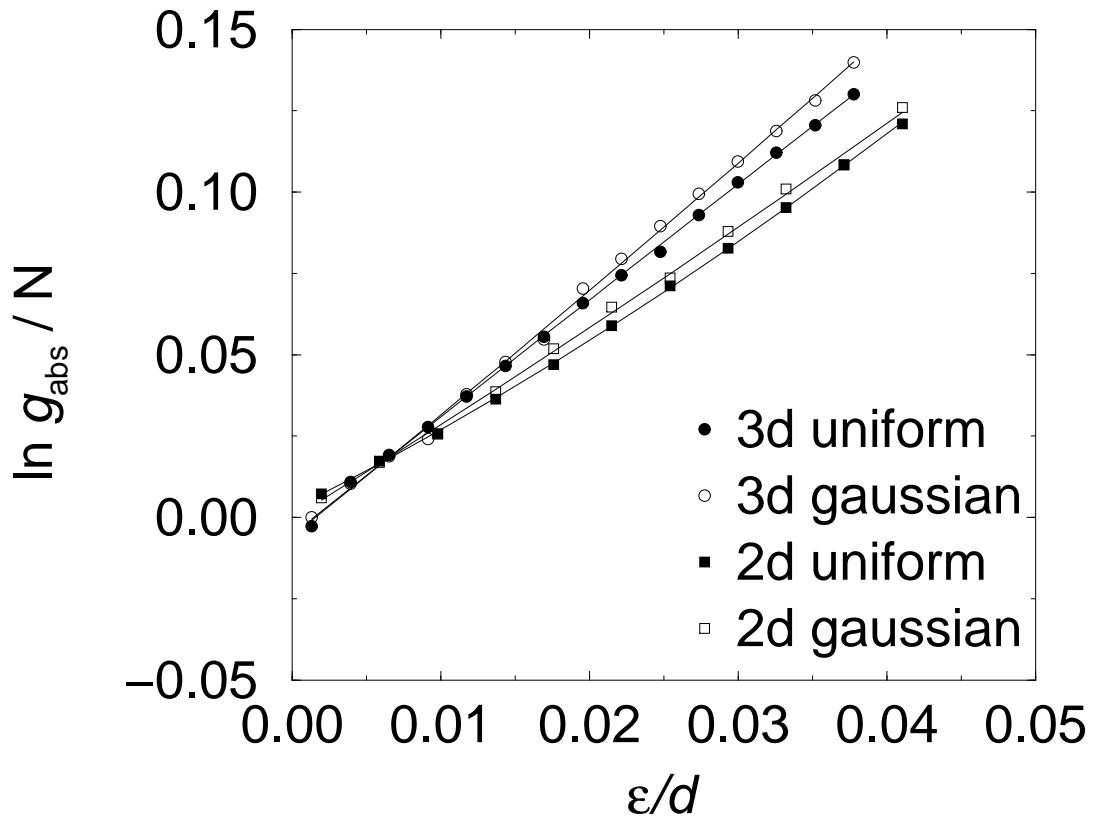


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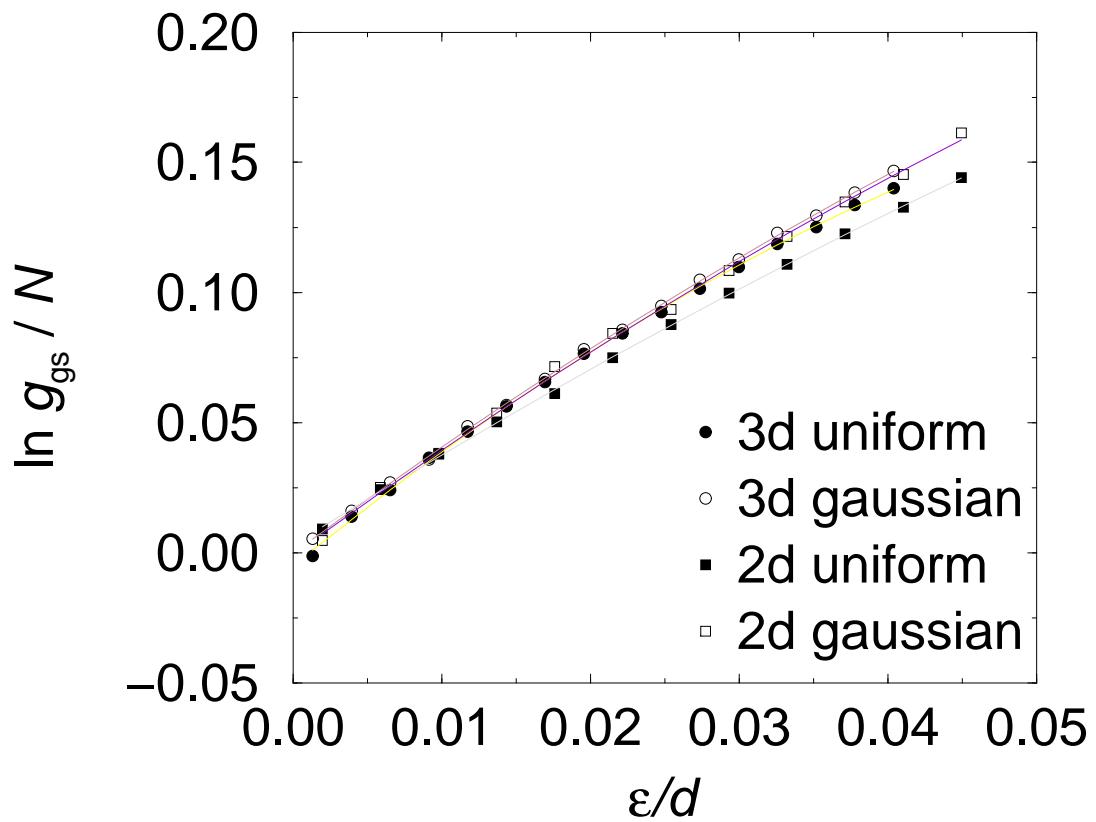


Figure 5: Local density of states in the ground-state valleys  $g_{\text{gs}}$  versus energy per bond  $\varepsilon/d$  for 2d (squares) and 3d (circles) systems with uniform (full symbols) or gaussian (open symbols) distribution of interactions averaged over different realizations of interactions and different valleys. The lines correspond to quadratic fits.

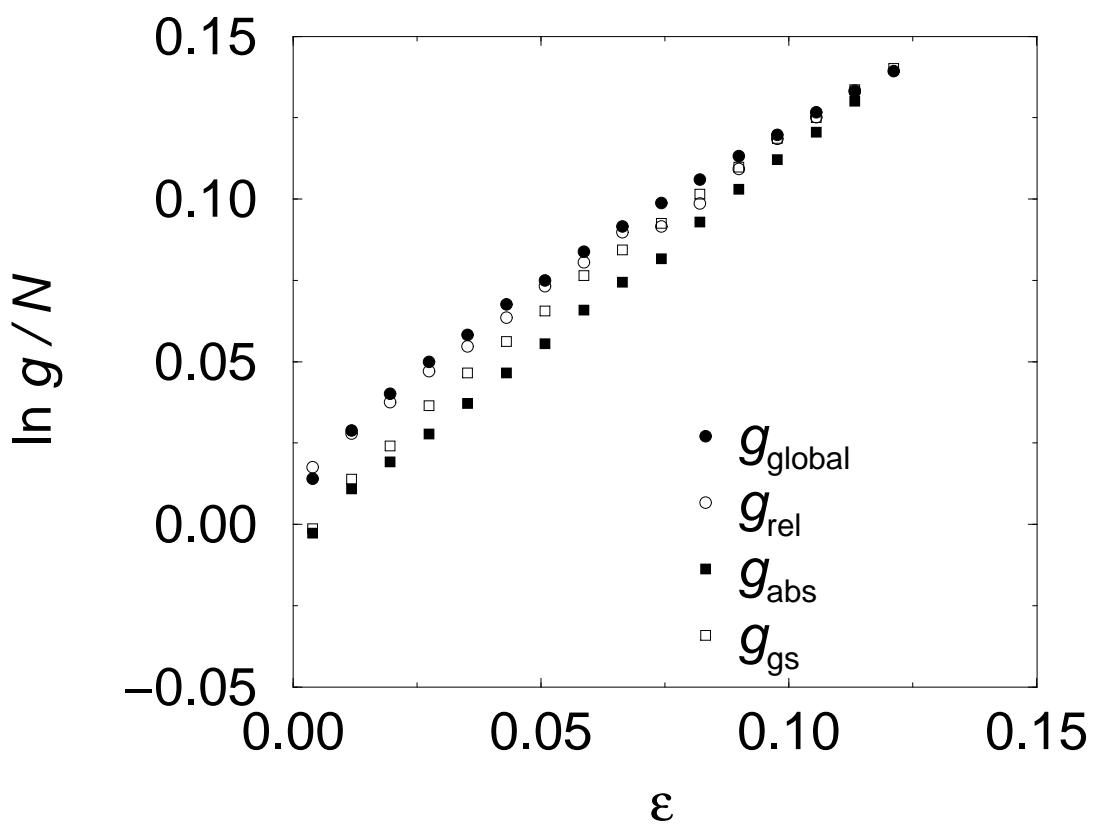


Figure 6: Comparison of the different DOS measures for the 3d uniform distributed case.